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Complex quasiperiodic self-similar tilings: their parameterization, boundaries, complexity, growth and symmetry

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A class of quasiperiodic tilings of the complex plane is discussed. These tilings are based on β-expansions corresponding to cubic irrationalities. There are three classes of tilings: Q_3, Q_4 and Q_5. These classes consist of three, four and five pairwise similar prototiles, respectively. A simple algorithm for construction of these tilings is considered. This algorithm uses greedy expansions of natural numbers on some sequence. Weak and strong parameterizations for tilings are obtained. Layerwise growth, the complexity function and the structure of fractal boundaries of tilings are studied. The parameterization of vertices and boundaries of tilings, and also similarity transformations of tilings, are considered.

1. Introduction

Using substitution dynamic systems, Rauzy (1982) constructed a complex quasiperiodic plane tiling consisting of three types of tiles with fractal boundaries. These tiles are pairwise similar. Moreover, they are self-similar. Recall that a self-similar tile is a compact set with non-empty interior that consists of a finite number of similar tiles. Thurston (1990) and Kenyon (1996) discovered self-similar quasiperiodic tilings invariant under a scaling symmetry with respect to expansion β. Another approach to the construction of self-similar tilings was suggested by Akiyama (1999), who used a β-expansion to construct tiles, where β is a Pisot number. Now the following methods of construction of self-similar tilings are known: substitution dynamic systems (Rauzy, 1982; Ito & Ohtsuki, 1993; Arnoux & Ito, 2001; Sirvent & Wang, 2002); numeration systems for natural numbers (Rauzy, 1982; Zhuravlev, 2006); β-expansions (Akiyama, 1999); digit tilings (Lagarias & Wang, 1996); radix representations (Akiyama *et al.*, 2005); and iterated function systems (Hutchinson, 1981).

In this paper we use the method of β-expansions corresponding to cubic irrationalities. There are three classes of tilings: Q_3, Q_4 and Q_5, which consist of three, four and five tile types, respectively. Tiles of these types are pairwise similar. We discuss a new approach to these tilings based on weak and strong parameterizations for tilings.

The given approach goes back to an old idea of de Wolff (1974) of using higher-dimensional spaces for the description of incommensurately modulated structures. This idea was extended to quasiperiodic tilings by de Bruijn (1981) and Janssen (1986). With such an approach we naturally have a physical space with the tiling and an internal space with parameters of tiles. There is symmetry between these spaces. Thus we can swap physical and internal spaces.

In our case we can consider the complex plane as the physical space and the real line as the internal space. It turns out that the language of parameters is very convenient for describing geometric properties of self-similar tilings: types of local encirclements, structure of boundaries, similarity transformations of the tilings and so on. This approach is a generalization of the well known concept of model sets [see, for example, Moody (2000)]. Rauzy points of the tiles (special points from tiles) and vertices of tilings are model sets.

Earlier this approach was used to study one special quasiperiodic tiling (Rauzy tiling). In particular, its layerwise growth (Zhuravlev & Maleev, 2007*a*, 2008*a*), complexity function and forcing (Zhuravlev & Maleev, 2007*b*), pure point diffraction (Zhuravlev & Maleev, 2008*b*), and structure and similarities of its fractal boundaries (Zhuravlev & Maleev, 2009*a*,*b*) were studied.

2. Construction of the quasiperiodic tiling

Consider the cubic equation

$$x^3 + px^2 + qx = 1 \tag{1}$$

with integer coefficients. Assume the following conditions on the coefficients,

$$\begin{aligned} &p + q \geq 1, \\ &\Delta(p, q) = 4p^3 + p^2q^2 - 4q^3 - 18pq - 27 < 0, \end{aligned} \tag{2}$$

$$-1 \leq p \leq q + 1. \tag{3}$$

The condition (2) is equivalent to the following property:

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(1) Equation (1) has a unique real root ζ from the segment [0, 1] and two complex roots β and $\tilde{\beta}$ with $|\beta| = |\tilde{\beta}| > 1$.

Akiyama (2000) proved that the condition (3) is equivalent to the following property:

(2) Consider an arbitrary $r \in [0, 1]$ such that $r \in \mathbb{Z}[\zeta]$. Then the expansion $x = \sum_{i=1}^{\infty} \varepsilon_i \zeta^i$, obtained by a greedy algorithm, is finite.

Recall that the expansion

$$r = \sum_{i=1}^{\infty} \varepsilon_i \zeta^i, \qquad \varepsilon_i \geq 0 \tag{4}$$

is greedy if, and only if, the inequality $|r - \sum_{i=1}^{m} \varepsilon_i \zeta^i| < \zeta^m$ holds for all m. It is known that the sequence $\{\varepsilon_i\}$ from expansion (4) is not arbitrary. It must satisfy some conditions determined by equation (1).

Now we describe these conditions. For this, we split the set Q of cubic equations with the conditions (2) and (3) on three sets Q_3, Q_4 and Q_5. Let Q_4 be a subset of Q consisting of equations with $p = -1$. If we use notation a instead of q, we can rewrite the equations from Q_4 as

$$x^3 - x^2 + ax = 1, \qquad a \in \mathbb{Z},\ a \geq 2.$$

Now let Q_5 be a subset of Q consisting of equations with $p = q + 1$. If we again use a instead of q, we can rewrite the equations from Q_5 as

$$x^3 + (a+1)x^2 + ax = 1, \qquad a \in \mathbb{Z},\ a \geq 0.$$

All remaining cubic equations, satisfying conditions (2) and (3), are equations from the set Q_3. So we can say that the set Q_3 consists of admissible cubic equations of general position. The sets Q_4 and Q_5 consist of codimension one 'degenerations'. Reasons for this subdivision of set Q will be explained below.

For any equation from Q_3 and any $i > 1$ coefficients $\{\varepsilon_i\}$ satisfy the inequality $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2} <_{\rm lex} qp1$. Here $<_{\rm lex}$ means that the sequence of numbers on the left of this sign is less that the sequence on the right in lexicographic order. Similarly, for two other sets of equations we can write the corresponding conditions on $\{\varepsilon_i\}$ as follows: $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3} <_{\rm lex} (a-1)(a-1)01$ for Q_4 and any $i > 2$, $\varepsilon_i\varepsilon_{i-1}\varepsilon_{i-2}\varepsilon_{i-3}\varepsilon_{i-4} <_{\rm lex} (a+1)00a1$ for Q_5 and any $i > 3$. In all cases it is necessary to put $\varepsilon_0 = 0$. These inequalities are well known [see, for example, Parry (1960)].

In Fig. 1 we denote by triangles the points (p, q) corresponding to the equations from Q_3. Similarly, squares correspond to the equations from Q_4 and pentagons to those from Q_5.

Now we consider all admissible greedy expansions (4) (not only for points from [0; 1]). To each expansion we put in correspondence a complex number $\sum_{i=1}^{\infty} \varepsilon_i \beta^{-i}$. Denote by A the set of the sequences $\{\varepsilon_i\}$ forming admissible expansions. Then we can define the plane point set

$$T = T(p, q) = \left\{ \sum_{i=1}^{\infty} \varepsilon_i \beta^{-i} : \{\varepsilon_i\}_{i=1}^{\infty} \in A \right\}. \tag{5}$$

The set (5) was defined by Akiyama (1999, 2000). It is known that $T(p, q)$ is a compact arcwise set with a fractal boundary. The point of origin is an inner point of this set. This set produces a self-similar plane tiling Til(p, q). Examples of these tilings are represented in Fig. 2. We will consider the algorithm for the construction of these tilings in detail.

To obtain a self-similar plane tiling, we must specify a partition of the figure $T(p, q)$ into the figures pairwise similar among themselves. We will say that these figures are tiles. Using described sets Q_k we can obtain these partitions as follows. Consider a cubic equation from the set Q_k. The figure $T(p, q)$ can be represented as a union of k pairwise similar tiles among themselves. The membership of the concrete point $z = \sum_{i=1}^{\infty} \varepsilon_i \beta^{-i} \in T(p, q)$ to one of k tiles is unambiguously

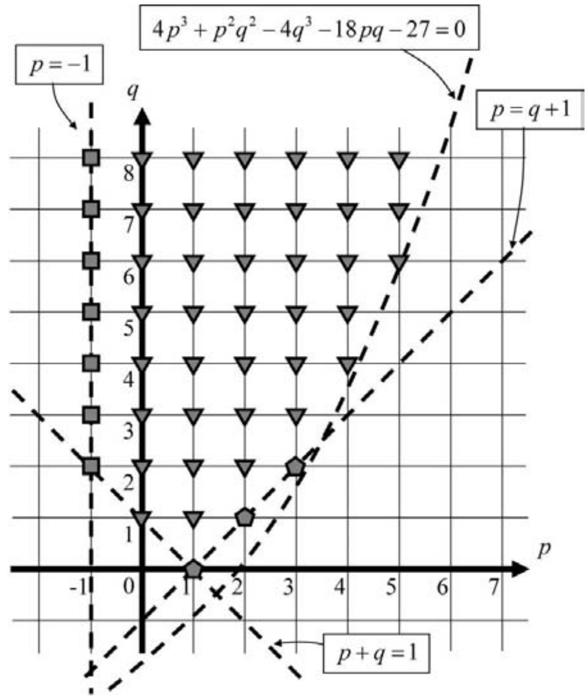

Figure 1
Parameters p and q of cubic equations from the sets Q_3, Q_4 and Q_4.

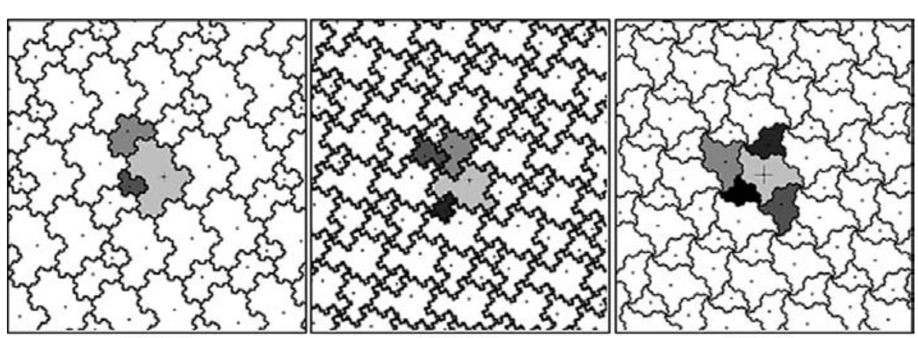

Figure 2
Fragments of self-similar plane tilings.

Table 1
Tuples $A_k^{(m)}$ determined by a partition of the figure $T(p, q)$ into the tiles $T^{(m)}(p, q)$.

		ε_1	ε_2	ε_3	ε_4
Q_3	$A_3^{(3)}$	p	q	–	–
	$A_3^{(2)}$	q	$0, 1, \ldots, q-1$	–	–
	$A_3^{(1)} = A_3/(A_3^{(2)} \cup A_3^{(3)}$			–	–
Q_4	$A_4^{(4)}$	0	$a-1$	$a-1$	–
	$A_4^{(3)}$	$a-1$	$a-1$	*	–
	$A_4^{(2)}$	$a-1$	$0, 1, \ldots, a-2$	*	–
	$A_4^{(1)} = A_4/(A_4^{(2)} \cup A_4^{(3)} \cup A_4^{(4)})$				–
Q_5	$A_5^{(5)}$	a	0	0	$a+1$
	$A_5^{(4)}$	0	0	$a+1$	*
	$A_5^{(3)}$	0	$a+1$	*	*
	$A_5^{(2)}$	$a+1$	*	*	*
	$A_5^{(1)} = A_5/(A_5^{(2)} \cup A_5^{(3)} \cup A_5^{(4)} \cup A_5^{(5)})$				

determined by the first $k-1$ digits (coefficients $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}$) of the expansion of z to the powers of β.

More precisely, let A_k be a set of tuples $(\varepsilon_1, \ldots, \varepsilon_{k-1})$ which can be continued to admissible expansions from A. Then A_k can be represented as a union of k non-intersecting sets $A_k^{(1)}, A_k^{(2)}, \ldots, A_k^{(k)}$. Corresponding tiles

$$T^{(m)}(p, q) = \left\{ \sum_{i=1}^{\infty} \varepsilon_i \beta^{-i} : \{\varepsilon_i\}_{i=1}^{\infty} \in A, (\varepsilon_i)_{i=1}^{k-1} \in A_k^{(m)} \right\}$$

are pairwise similar. These tiles form the required partition of the $T(p, q)$. Tuples $A_k^{(m)}$ for sets Q_k are represented in Table 1. In this table a dash (–) means that these coefficients are not used in the definition of the set A_k. Similarly, an asterisk (*) means that we can use any coefficients which are possible in the greedy expansions for this tiling.

The described partition of $T(p, q)$ produces a self-similar quasiperiodic plane tiling on k tile types.

We will say that partition

$$\beta^k T(p, q) = \bigcup_{m=1}^{k} \beta^k T^{(m)}(p, q)$$

is a tiling $\mathrm{Til}_0(p, q)$ of level zero. Now, the set of the type m tiles from the tiling $\mathrm{Til}_n(p, q)$ of level n is

$$\mathrm{Til}_n^{(m)}(p, q) = \beta^{n+k} \left\{ \sum_{i=1}^{\infty} \varepsilon_i \beta^{-i} : \{\varepsilon_i\}_{i=1}^{\infty} \in A, (\varepsilon_i)_{i=n+1}^{n+k-1} \in A_k^{(m)} \right\}.$$

Note that the tiling $\mathrm{Til}_0(p, q)$ contains the point of origin. Moreover, the tiling $\mathrm{Til}_n(p, q)$ is a patch in the tiling $\mathrm{Til}_{n+1}(p, q)$. So, in the limit $n \to \infty$ we have a planar quasiperiodic tiling $\mathrm{Til}(p, q)$. This tiling consists of tiles of k types. Any two tiles of different types are similar to each other.

This way of constructing tiles $T^{(m)}(p, q)$ and the corresponding tiling is not suitable for computer realization. It is connected to the infinity of the sequences from A and to the difficulty of processing their enumeration. Therefore, we offer a modified algorithm for this problem.

For every cubic equation with the conditions (2) and (3) we construct a sequence $\{t_n\}$ defined by a following recurrent relation and initial conditions,

$$\begin{aligned} &t_{i+1} = qt_i + pt_{i-1} + t_{i-2}, t_0 = 1, t_1 = 1, t_2 = q+1 \text{ for } Q_3, \\ &t_{i+1} = at_i - t_{i-1} + t_{i-2}, t_0 = 1, t_1 = 1, t_2 = a \text{ for } Q_4, \text{ and} \\ &t_{i+1} = (a+1)t_i + at_{i-3} + t_{i-4}, t_0 = 1, t_1 = 1, t_2 = a+1, \\ &\quad t_3 = a^2 + 3a + 3, t_4 = a^3 + 4a^2 + 6a + 4 \text{ for } Q_5. \end{aligned}$$

Note that t_n is the total number of admissible greedy expansions of finite length $n-1$.

For any non-negative integer number l we can obtain the greedy expansion on this sequence: $l = \sum_{i=1}^{h=h(l)} \varepsilon_i(l) t_i$, with $0 \le l - \Sigma_{i=1}^{g} \varepsilon_i(l) t_i < t_g$ for any $g \le h$.

Accordingly, every such expansion determines a complex number (plane point) $z(l) = \sum_{i=1}^{h(l)} \varepsilon_i(l) \beta^{-i}$.

Proposition 1. The figure $T(p, q)$ is a closure of the set of all such points $z(l)$:

$$T(p, q) = \overline{\{z(l) : l \in \mathbb{N}\}}. \qquad (6)$$

Formula (6) actually gives a new algorithm of construction of the figure $T(p, q)$. This algorithm is much more convenient for computer realization. In Figs. 3, 4 and 5 we represent some examples of the figures $T(p, q)$ consisting of the tiles $T^{(m)}(p, q)$ for the sets Q_3, Q_4 and Q_5.

Similarly, the formula for $\mathrm{Til}_n^{(m)}(p, q)$ can be written as

$$\mathrm{Til}_n^{(m)}(p, q) = \beta^{n+k} \overline{\{z(l) : l \in \mathbb{N}, (\varepsilon_{n+1}(l), \ldots, \varepsilon_{n+k-1}(l)) \in A_k^{(m)}\}}.$$

Thus we have a new computer algorithm for the construction of the tiling.

3. Weak parameterization

The approach to construction of the tilings $\mathrm{Til}(p, q)$ described above can be used for computer plotting of tilings, but it has two limitations: (i) the high computational complexity of the algorithm; and (ii) the inconvenience for further research of the tiling.

The papers by Zhuravlev & Maleev (2007*a*, 2008*a*) and Shutov & Maleev (2008) for Rauzy and Ito–Ohtsuki tilings offered an alternative approach to the construction and studying of quasiperiodic tilings. This approach is based on parameterizations.

Weak parameterization for a tiling gives one-to-one correspondence between tiles and parameters from some one-dimensional set. To obtain weak parameterization for the tilings $\mathrm{Til}(p, q)$ let us introduce the following definition.

Consider a similarity transformation which maps the tile with the point of origin to some fixed tile T. The image of the point of origin under this transformation is called a Rauzy point of the tile T.

Note that the corresponding similarity maps have the form $z \to \beta^n z + \alpha$ with $\alpha \in \mathbb{Z}[\beta]$. Therefore any Rauzy point belongs to the ring $\mathbb{Z}[\beta]$. Since β is a root of a cubic equation, all Rauzy points can be represented in the form $c_0 + c_1\beta + c_2\beta^2$.

Define the operator of conjugation ($'$):

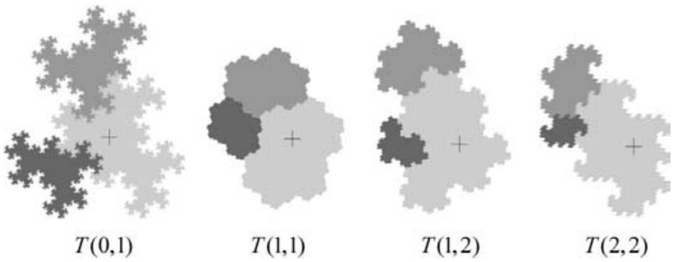

Figure 3
Examples of $T(p,q)$ consisting of tiles $T^{(m)}(p,q)$ for the set Q_3.

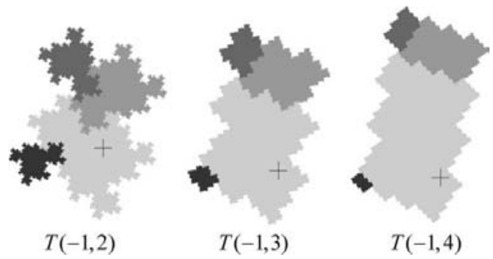

Figure 4
Examples of $T(p,q)$ consisting of tiles $T^{(m)}(p,q)$ for the set Q_4.

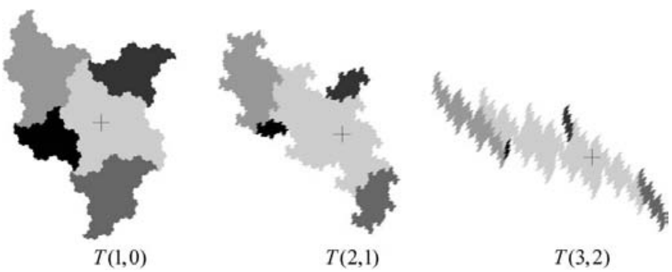

Figure 5
All possibilities for the set of Q_5 figures $T(p,q)$ consisting of tiles $T^{(m)}(p,q)$.

$$(c_0 + c_1\beta + c_2\beta^2)' = c_0 + c_1\zeta + c_2\zeta^2,$$

$$(c_0 + c_1\zeta + c_2\zeta^2)' = c_0 + c_1\beta + c_2\beta^2.$$

This operator is the algebraic conjugation in the ring generated by three roots of a cubic equation. Also this operator is a bijection between rings $\mathbb{Z}[\beta]$ and $\mathbb{Z}[\zeta]$.

Denote by $R(p,q)$ the set of all Rauzy points of the tiling. Let $R^{(m)}(p,q)$ be a set of Rauzy points of type m tiles. Then $I(p,q) = R(p,q)'$ and $I^{(m)}(p,q) = R^{(m)}(p,q)'$ are corresponding parameter sets.

Proposition 2. The set $I^{(m)}(p,q)$ is an intersection of the ring $\mathbb{Z}[\zeta]$ with some right-open interval. Moreover, $I^{(m)}(p,q) \subset [0;1)$.

Corollary. The closure $\overline{I^{(m)}(p,q)}$ is the same segment. The closure $\overline{I(p,q)}$ is a union of a finite number of segments. It is a compact set.

For mathematical construction of a weak parameterization it is possible to use the following method. Let $A_k^{(m)}$ be the set of k-tuples which determines the tile type. Let $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \ldots, \overline{\varepsilon}_{k-1}$ be the least (in the lexicographic order) tuple from $A_k^{(m)}$. Then

Table 2
Weak parameterization for the tilings.

Set	m	$\#\mathrm{Til}_n^{(m)}$	Left end $\delta_1^{(m)}$	Length $l^{(m)}$	Right end $\delta_2^{(m)}$
Q_3	1	t_{n+1}	0	ζ^2	ζ^2
	2	$t_{n+1} + pt_n$	$q\zeta^2$	$\zeta^4 + p\zeta^3$	ζ
	3	t_n	$p\zeta^2 + q\zeta$	ζ^3	1
Q_4	1	t_{n+1}	0	ζ^3	ζ^3
	2	$t_{n+1} - t_n + t_{n+1}$	$(a-1)\zeta^3$	$\zeta^5 - \zeta^4 + \zeta^3$	ζ^2
	3	t_{n-1}	$(a-1)(\zeta^3 + \zeta^2)$	ζ^5	ζ
	4	t_n	$(a-1)(\zeta^2 + \zeta)$	ζ^4	1
Q_5	1	t_{n+1}	0	ζ^4	ζ^4
	2	$t_{n-3} + at_{n-2}$	$(a+1)\zeta^4$	$\zeta^8 + a\zeta^7$	ζ^3
	3	$t_{n-2} + at_{n-1}$	$(a+1)\zeta^3$	$\zeta^7 + a\zeta^6$	ζ^2
	4	$t_{n-1} + at_n$	$(a+1)\zeta^2$	$\zeta^6 + a\zeta^5$	ζ
	5	t_n	$(a\zeta^4 + (a+1)\zeta$	ζ^5	1

$$\delta_1^{(m)} = \lim_{n\to\infty} \frac{(\overline{\varepsilon}_1 t_{n+1} + \overline{\varepsilon}_2 t_{n+2} + \ldots + \overline{\varepsilon}_{k-1} t_{n+k-1})}{t_{n+k}}$$

is the left end of the segment $\overline{I^{(m)}(p,q)}$. Considering the asymptotic

$$t_n \simeq c(p,q)\zeta^{-n}, \tag{7}$$

we find

$$\delta_1^{(m)} = \overline{\varepsilon}_1\zeta^{k-1} + \overline{\varepsilon}_2\zeta^{k-2} + \ldots + \overline{\varepsilon}_{k-1}\zeta. \tag{8}$$

Using Table 1 and formula (8) we can obtain all left ends $\delta_1^{(m)}$ of the segments $\overline{I^{(m)}(p,q)}$.

The length of the segment $\overline{I^{(m)}(p,q)}$ is calculated using the formula

$$l^{(m)} = \lim_{n\to\infty} \frac{\#\mathrm{Til}_n^{(m)}}{t_{n+k}},$$

where $\#\mathrm{Til}_n^{(m)}$ is the number of type m tiles in the patch $\mathrm{Til}_n(p,q)$. For practical calculation of $l^{(m)}$ we can express $\#\mathrm{Til}_n^{(m)}$ through t_n and use asymptotic (7).

So, for example, for the set Q_3 and any $n \geq 1$ we have $\#\mathrm{Til}_n^{(1)} = t_{n+1}$, $\#\mathrm{Til}_n^{(2)} = t_{n-1} + pt_n$; $\#\mathrm{Til}_n^{(3)} = t_n$. Therefore

$$l^{(1)} = \lim_{n\to\infty} \frac{t_{n+1}}{t_{n+3}} = \zeta^2,$$

$$l^{(2)} = \lim_{n\to\infty} \frac{t_{n-1} + pt_n}{t_{n+3}} = \zeta^4 + p\zeta^3,$$

$$l^{(3)} = \lim_{n\to\infty} \frac{t_n}{t_{n+3}} = \zeta^3.$$

The lengths of the intervals $l^{(m)}$ for the sets Q_4 and Q_5 are calculated similarly. Expressions of $\#\mathrm{Til}_n^{(m)}$ through t_n, left ends $\delta_1^{(m)}$, lengths $l^{(m)}$ and right ends $\delta_2^{(m)}$ of the intervals for all sets are represented in Table 2.

The existence of weak parameterization implies that the set of Rauzy points $R(p,q)$ is a model set.

Model sets are a well known mathematical model of quasicrystals [see, for example, Meyer (1972); Moody (2000)]. There is a mathematical theory of model sets which can be applied to the sets $R(p,q)$. From this theory in particular it follows that the set of Rauzy points $R(p,q)$ has a pure point diffraction (Schlottmann, 1998, 2000; Zhuravlev & Maleev, 2008*b*). A similar property holds for the sets $R^{(m)}(p,q)$ too.

Now we introduce an important definition. The nuclear $\mathrm{Nucl}(p, q)$ of the tiling $\mathrm{Til}(p, q)$ is a union of tiles whose Rauzy points have parameters $\delta_1^{(m)}$, that is the left ends of intervals of weak parameterization.

The nuclear has the following elementary properties:

(i) The nuclear $\mathrm{Nucl}(p, q)$ contains exactly one tile for each tile type.

(ii) The tile containing the point of origin belongs to the nuclear.

(iii) The nuclear $\mathrm{Nucl}(p, q)$ coincides with tiling $\mathrm{Til}_0(p, q)$ of level zero.

The fundamental role of the nuclear $\mathrm{Nucl}(p, q)$ in the study of the tilings $\mathrm{Til}(p, q)$ will be shown below.

4. Strong parameterization for the tiling

Weak parameterization for tilings described above does not consider the neighborhood of the tiles. So we need another type of parameterization. Let us call it strong parameterization.

Two tiles from $\mathrm{Til}(p, q)$ are called neighboring if they have a common part of boundary. Let $x \in R(p, q)$ be a Rauzy point of some tile T. The local star $S(x)$ is a set of vectors traced from this Rauzy point x to Rauzy points of tiles neighboring T. Obviously the vectors from $S(x)$ belong to the ring $\mathbb{Z}[\beta]$ (as all Rauzy points have coordinates from $\mathbb{Z}[\beta]$). Therefore, the conjugation operator maps the set $S(x)$ to the set of local numbers $S'(x')$. The local numbers are defined by the following property. If x' is a parameter of some tile, then $\{x' + y' : y' \in S'(x')\}$ is the set of parameters of the tiles neighboring this tile. Strong parameterization is a description of sets of local numbers $S'(x')$ for parameters x'. It is more convenient to consider local color stars $\mathrm{CS}(x)$ in which each local vector has weight, the number equal to the type of the neighboring tile. Denote by $\mathrm{CS}'(x')$ corresponding to $\mathrm{CS}(x)$ the set of local color numbers of the parameter x'.

Proposition 3. The tiling $\mathrm{Til}(p, q)$ has only finite types of local color stars $\mathrm{CS}(x)$.

Assume that the tiling $\mathrm{Til}(p, q)$ has only r types of local color stars. Denote by $\hat{R}^{(i)}(p, q)$ the set of Rauzy points with local color star of type i ($i = 1, 2, \ldots, r$). Let $\hat{I}^{(i)}(p, q) = \hat{R}^{(i)}(p, q)'$ be the corresponding parameter sets.

Proposition 4. $\hat{I}^{(i)}(p, q) \subset I(p, q)$ is an intersection of the ring $\mathbb{Z}[\zeta]$ with some right-open interval.

So, construction of strong parameterization for the tiling $\mathrm{Til}(p, q)$ demands determination of the intervals $\hat{I}^{(i)}(p, q)$, $i = 1, 2, \ldots, r$ and calculation of corresponding sets of local numbers. In Fig. 6 we represent 11 types of local color stars $\mathrm{CS}(x)$ of the tiling $\mathrm{Til}(1, 2)$ and corresponding intervals of parameters $\hat{I}^{(i)}(1, 2)$. In Table 3 and 4 values of local numbers and strong parameterization for the tiling $\mathrm{Til}(1, 2)$ are represented.

The constructed strong parameterization can be used, in particular, for the modeling of layerwise growth for the tilings $\mathrm{Til}(p, q)$. The geometrical model of layerwise growth was firstly introduced by Rau *et al.* (2002) and Zhuravlev (2002). In the sequel, the layerwise growth for various types of tilings was studied by Zhuravlev *et al.* (2002), Zhuravlev (2003), Shutov (2003) and Zhuravlev & Maleev (2007*a*, 2008*a*).

The layerwise growth can be defined as follows. Consider as a seed an arbitrary finite set P of tiles from $\mathrm{Til}(p, q)$. Tiles neighboring the tiles from P, with the exception of tiles from P, form the first coordination encirclement $eq_1(P)$. Tiles neighboring the tiles from $eq_1(P)$, with the exception of tiles from P and $eq_1(P)$, form the second coordination encirclement $eq_2(P)$. Repeating this process, we can receive nth coordination encirclement $eq_n(P)$ *etc.*

Computer modeling of layerwise growth with use of the strong parameterization gives the following result.

Conjecture 1. Tilings $\mathrm{Til}(p, q)$ have polygonal growth. More precisely, for every tiling there exists a convex centrally symmetric polygon $pol(p, q)$ such that $\lim_{n\to\infty} eq_n(P)/n = \mathrm{pol}(p, q)$.

Table 3
Local numbers for the tiling $\mathrm{Til}(1, 2)$.

Local number	Value
s_1	ζ^4
s_2	ζ^3
s_3	$\zeta^3 + \zeta^4$
s_4	$\zeta^2 - \zeta^3$
s_5	$\zeta^2 + \zeta^4$
s_6	$\zeta^2 + \zeta^3$
s_7	$\zeta - \zeta^2$
s_8	$2\zeta^2$
s_9	$2\zeta^2 + \zeta^4$
s_{10}	$2\zeta^2 + \zeta^3$
s_{11}	$1 - \zeta$
s_{12}	$1 - \zeta^2$
s_{13}	$2\zeta + \zeta^2$

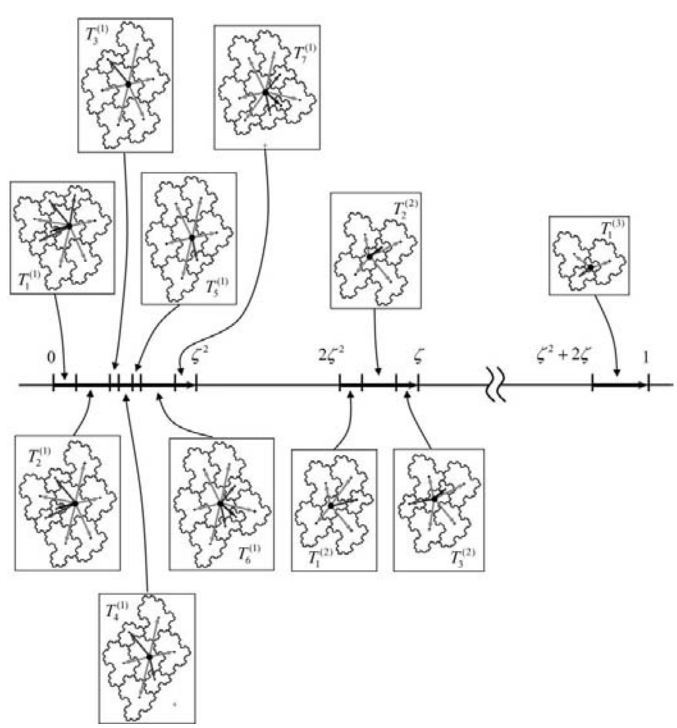

Figure 6
Eleven types of color local stars of the tiling $\mathrm{Til}(1, 2)$.

Table 4
Strong parameterization for the tiling Til(1, 2).

Local color star type i	Tile type $T_n^{(m)}$	Interval $\hat{I}^{(i)}(1,2)$	Number of neighboring tiles	Local numbers Type of the neighboring tile $m=1$	$m=2$	$m=3$
1	$T_1^{(1)}$	$[0, \zeta^4)$	8	s_1, s_2, s_3, s_4	s_8, s_9, s_{10}	s_{13}
2	$T_2^{(1)}$	$[\zeta^4, \zeta^3)$	8	$s_1, s_2, s_3, s_4, -s_1$	s_8, s_9	s_{13}
3	$T_3^{(1)}$	$[\zeta^3, \zeta^3+\zeta^5)$	6	$s_1, s_2, s_3, -s_1, -s_2$	s_8	
4	$T_4^{(1)}$	$[\zeta^3+\zeta^5, \zeta^3+\zeta^4)$	6	$s_1, s_2, -s_1, -s_2$	s_7, s_8	
5	$T_5^{(1)}$	$[\zeta^3+\zeta^4, \zeta^2-\zeta^3)$	6	$s_1, s_2, -s_1, -s_2, -s_3$	s_7	
6	$T_6^{(1)}$	$[\zeta^2-\zeta^3, \zeta^2-\zeta^4)$	8	$s_1, -s_1, -s_2, -s_3, -s_4$	s_6, s_7	
7	$T_7^{(1)}$	$[\zeta^2-\zeta^4, \zeta^2)$	8	$-s_1, -s_2, -s_3, -s_4$	s_5, s_6, s_7	s_{12}
8	$T_1^{(2)}$	$[2\zeta^2, 2\zeta^2+\zeta^4)$	5	$-s_5, -s_6, -s_7, -s_8$	s_2	
9	$T_2^{(2)}$	$[2\zeta^2+\zeta^4, 2\zeta^2+\zeta^3)$	5	$-s_6, -s_7, -s_8, -s_9$		s_{11}
10	$T_3^{(2)}$	$[2\zeta^2+\zeta^3, \zeta)$	6	$-s_7, -s_8, -s_9, -s_{10}$	$-s_2$	s_{11}
11	$T_1^{(3)}$	$[2\zeta+\zeta^2, 1)$	3	$-s_{12}, -s_{13}$	$-s_{11}$	

In Fig. 7 we represent the layerwise growth process and the growth polygon for the tiling Til(1, 2).

5. Complexity function

By definition, the n-crown of the tile X from the tiling Til(p, q) is a union of its first n coordination encirclements:

$$C_n(X) = X \cup \bigcup_{i=1}^{n} eq_i(X).$$

The crown $C_n(X)$ can also be defined as a set of all tiles of the tiling with distance from the tile X not greater than n [in natural metric of the tiling, defined by Zhuravlev (2002)]. Two crowns $C_n(X)$ and $C_n(Y)$ are called equivalent if there exists a vector z such that $C_n(X) = C_n(Y) + z$. Let $N_{\mathrm{Til}(p,q)}(n)$ be a number of equivalence classes of n-crowns. The function $N_{\mathrm{Til}(p,q)}(n)$ is called the complexity function of the tiling Til(p, q).

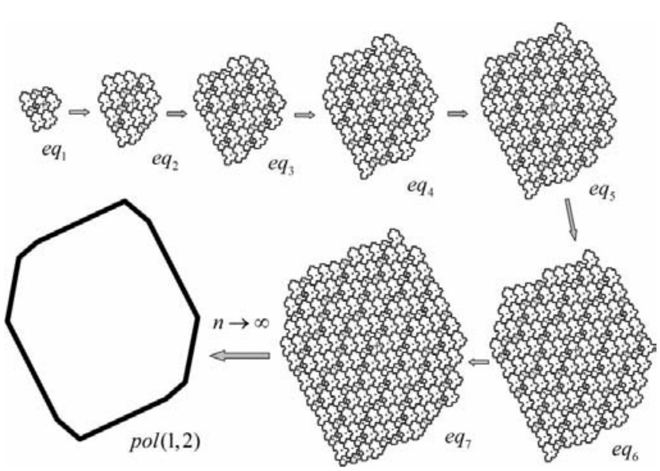

Figure 7
Layerwise growth for the tiling Til(1, 2).

It is obvious that for periodic tilings Til, $N_{\mathrm{Til}}(n)$ = constant for $n \geq n_0(\mathrm{Til})$. On the other hand, for a completely random tiling Til, the function $N_{\mathrm{Til}}(n)$ has at least exponential growth, or is even equal to infinity for any n. Study of the function $N_{\mathrm{Til}}(n)$ motivated, in particular, the well known conjecture connecting the order of growth of the function $N_{\mathrm{Til}}(n)$ with the structure of the translation group of the tiling Til (Pleasants, 2000).

In the paper by Zhuravlev & Maleev (2007*b*) the fundamental theorem about the complexity function of the tilings with strong parameterizations was proved. It is possible to check that the tilings Til(p, q) satisfy the conditions of this theorem. Thus, we have the following results connecting complexity function, layerwise growth and strong parameterization.

Proposition 5. In the tiling Til(p, q), the value of the complexity function $N_{\mathrm{Til}(p,q)}(n)$ is equal to the number of tiles in n-crown $C_n[\mathrm{Nucl}(p, q)]$, where Nucl(p, q) is the nuclear of tiling.

Proposition 6. Various tiles from $C_n[\mathrm{Nucl}(p, q)]$ have various n-crowns.

Proposition 7. Let $I_{n,X}$ be a set of parameters of tiles from Til(p, q), with n-crown equivalent to $C_n(X)$. Then $I_{n,X}$ is an intersection of the ring with some right-open interval. The left end of this interval is a parameter of the tile from $C_n[\mathrm{Nucl}(p, q)]$, with n-crown equivalent to $C_n(X)$.

Corollary. $N_{\mathrm{Til}(p,q)}(n) \simeq c_1(p, q)n^2$, where $c_1(p, q)$ is a constant depending only on p and q.

Proven results can be used to calculate strong parameterizations for the tilings. From proposition 7 it follows that the ends of the intervals $\hat{I}^{(i)}(p, q)$, $i = 1, 2, \ldots, r$ are parameters of tiles from $C_1[\mathrm{Nucl}(p, q)]$, and r is the number of tiles in $C_1[\mathrm{Nucl}(p, q)]$. Moreover, proposition 6 implies that for the determination of local color numbers it is sufficient to calculate corresponding local color vectors for all tiles from $C_1[\mathrm{Nucl}(p, q)]$.

In Fig. 8 a patch of the tiling Til(1, 2) is represented. The bold line selects 1-crown $C_1[\mathrm{Nucl}(1, 2)]$ of the nuclear Nucl(1, 2). Each of eleven tiles from $C_1[\mathrm{Nucl}(1, 2)]$ determines one of the possible local encirclements in the tiling Til(1, 2).

6. Vertices of the tiling and their parameters

If a point belongs at least to three tiles from Til(p, q) it is called a vertex of the tiling. Consider the problem of parameterization of vertices.

Proposition 8. Let v be a vertex of the tiling Til(p, q). Then $v \in \mathbb{Q}[\beta]$, *i.e.* $v = a + b\beta + c\beta^2$ with rational a, b and c.

The conjugation map $a + b\beta + c\beta^2 \leftrightarrow a + b\zeta + c\zeta^2$ can be continued from the ring $\mathbb{Z}[\beta]$ to the field $\mathbb{Q}[\beta]$. Thus we can consider a set of parameters of vertices $W(p, q) = V(p, q)'$, where $V(p, q)$ is the set of vertices of the tiling Til(p, q).

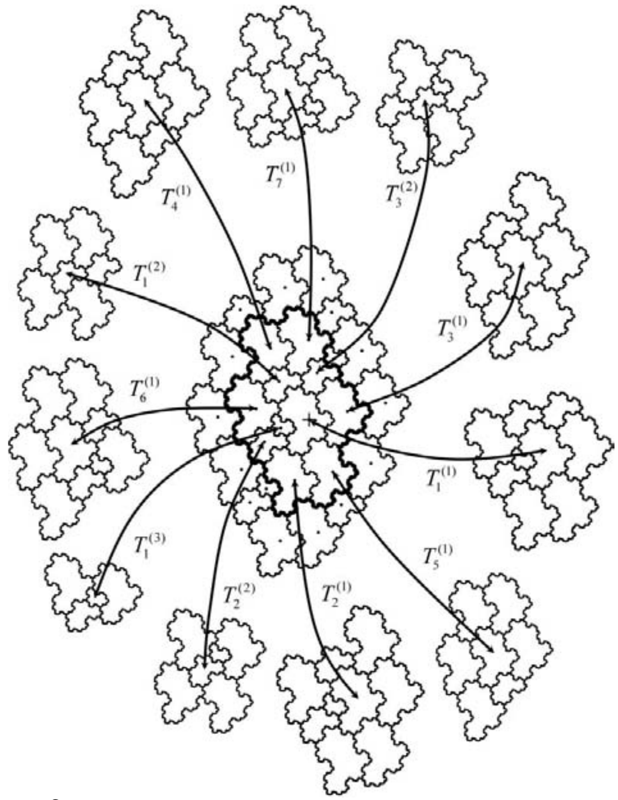

Figure 8
1-Crown $C_1[\mathrm{Nucl}(1,2)]$ of tiling $\mathrm{Til}(1,2)$ determining all possible local encirclements.

Let $V_0(p,q)$ be a set of vertices of the tiles from the 1-crown of nuclear $C_1[\mathrm{Nucl}(p,q)]$. Then we can calculate a set of parameters of vertices using the following algorithm.

Algorithm 1.

(i) Calculate parameters of all vertices from $V_0(p,q)$.

(ii) If the vertex $v_0 \in V_0(p,q)$ belongs to a tile with local color star of type i, add to sets of parameters the set $v_0' + \hat{I}^{(i)}(p,q)$. [Recall that $\hat{I}^{(i)}(p,q)$ is a set of parameters of Rauzy points with local color star of type i.]

(iii) Repeat step (ii) for every tile with vertex v_0.

(iv) Repeat steps (ii) and (iii) for every vertex $v_0 \in V_0(p,q)$.

The method of calculation of parameters of vertices from $V_0(p,q)$ can be found in the papers of Akiyama & Sadahiro (1998) and Zhuravlev & Maleev (2009*b*).

From the algorithm described above we have the following result.

Proposition 9. The set of parameters of vertices $W(p,q)$ is a finite union of non-intersecting sets

$$W(p,q) = \bigcup_{i=1}^{t_v} W^{(i)}(p,q),$$

where $W^{(i)}(p,q)$ is an intersection of some right-open interval and the set $\zeta^{(v_i)} + \mathbb{Z}[\zeta]$, and $\zeta^{(v_i)} \in \mathbb{Q}[\zeta]$.

It is obvious that we can take $\zeta^{(v_i)} \in W_0(p,q)$, and $t_v \le \operatorname{card} V_0(p,q)$.

Corollary. The set of vertices of the tiling $\mathrm{Til}(p,q)$ is a model set.

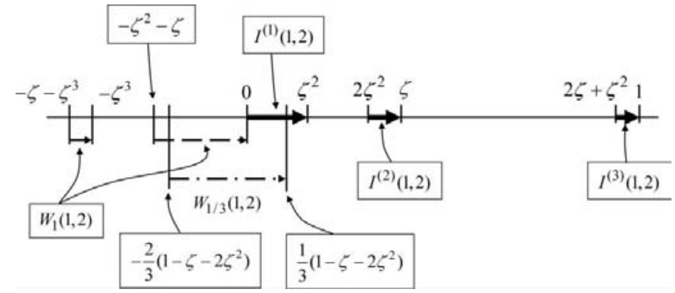

Figure 9
Parameters of vertices of the tiling $\mathrm{Til}(1,2)$.

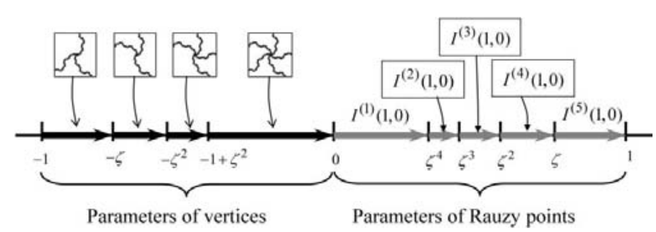

Figure 10
Parameters of vertices of the tiling $\mathrm{Til}(1,0)$.

In Fig. 9 parameters of vertices of the tiling $\mathrm{Til}(1,2)$ are represented. We can see two intervals with integer parameters (*i.e.* with parameters from $\mathbb{Z}[\zeta]$) $W_1(1,2)$ and interval $W_{1/3}(1,2)$ with fractional parameters of the form $[(1+2\zeta+\zeta^2)/3] + \mathbb{Z}[\zeta]$.

Now suppose that there are some different local types of vertices in the tiling. In this case we can correspond to every local type of vertices a finite number of intervals from the parameter set $W(p,q)$. In other words, different local types of vertices can be parameterized. So, in the tiling $\mathrm{Til}(1,2)$ there are two different local types of vertices. The first type corresponds to parameters from the set $W_1(1,2)$, the second to the set $W_{1/3}(1,2)$.

As another example, consider the parameterization of vertices of the tiling $\mathrm{Til}(1,0)$. In this case there are four different local types of vertices. All vertices have integer parameters. Each local type of vertices corresponds to one interval from the parameter set $W(1,0)$ (see Fig. 10).

7. Fractal boundaries

Now we consider boundaries of the tiling $\mathrm{Til}(p,q)$. Let $\Gamma(p,q)$ be a boundary of the tile containing the point of origin.

Conjecture 2. The boundary $\Gamma(p,q)$ can be represented as a union of non-intersected connected sets

$$\Gamma(p,q) = \bigcup_{i=0}^{t_\gamma} \gamma_i(p,q) \tag{9}$$

with the following property: for $i = 1, 2, \ldots, t_\gamma$ there exist similarity transformations h_i with factors β^{k_i}, such that

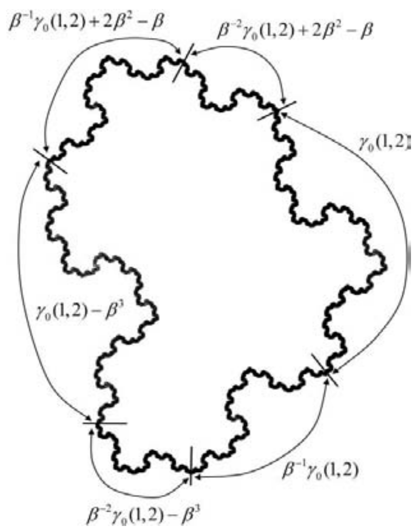

Figure 11
Boundary of the tile of the tiling Til(1, 2), split into elementary boundaries.

$$\gamma_i(p, q) = h_i[\gamma_0(p, q)]. \tag{10}$$

The sets $\gamma_i(p, q)$ are called elementary boundaries.

Fig. 11 shows the tile of the tiling Til(1, 2) containing the point of origin. The boundary is split into the elementary boundaries similar to $\gamma_0(1, 2)$.

Thus, the study of the boundary of the tile is reduced to the study of the set $\gamma_0(p, q)$. There exists an elementary recurrent construction of the set $\gamma_0(p, q)$. This construction is like the construction of some classic fractals (such as the von Koch curve).

A polyline with oriented segments is called an oriented polyline. In other words, an oriented polyline is a union of directed segments such that corresponding non-directed segments form a polyline.

Let L be an oriented polyline. Define a map constr_L of the set of all oriented polylines to itself as follows. Let A, B be the start and finish points of the polyline L. Let $\overrightarrow{A_1A_2}$ be an arbitrary directed segment. Let h be a similarity transformation map $\overrightarrow{A_1A_2}$ to $\overrightarrow{AB}$. Then

$$\mathrm{constr}_L(\overrightarrow{A_1A_2}) = h^{-1} \circ \mathrm{constr}_L \circ h(\overrightarrow{A_1A_2}) = h^{-1}(L).$$

For an arbitrary oriented polyline L_1, we denote by $\mathrm{constr}_L(L_1)$ the union of all images of its oriented segments under the map constr_L. Denote by $\mathrm{constr}_L^{(n)}$ the nth iteration of the map constr_L.

Conjecture 3. Suppose that Z_0 and Z_1 are the start and finish points of the curve $\gamma_0(p, q)$. Then there exists an oriented polyline L such that

$$\gamma_0(p, q) = \lim_{n\to\infty} \mathrm{constr}_L^{(n)}(\overrightarrow{Z_0Z_1}). \tag{11}$$

Moreover, all vertices of the polyline $\mathrm{constr}_L^{(n)}(\overrightarrow{Z_0Z_1})$ belong to $\gamma_0(p, q)$.

In Fig. 12, an oriented polyline L and its images under the first four iterations of the map constr_L are represented.

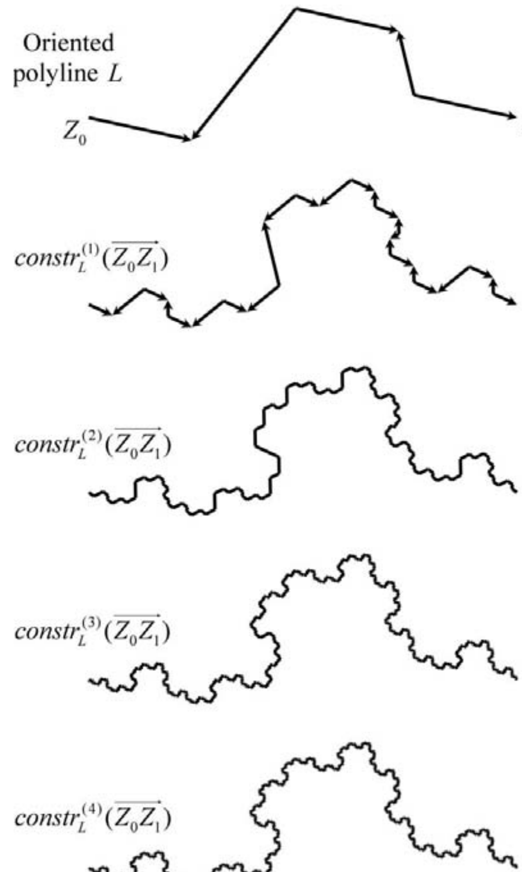

Figure 12
Polyline L and its images under the first four iterations of the map constr_L.

In spite of equation (11),

$$\gamma_0(p, q) \neq \bigcup_{n=1}^{\infty} \mathrm{constr}_L^{(n)}(\overrightarrow{Z_0Z_1}),$$

i.e. not all points of boundary $\gamma_0(p, q)$ can be constructed using the map constr_L.

Corollary. The boundary $\Gamma(p, q)$ is a fractal set. Its Hausdorf dimension is greater than one.

For practical calculation of map constr_L we can use the following conjecture.

Conjecture 4. Suppose that the vertices of the polyline $\mathrm{constr}_L(\overrightarrow{Z_0Z_1})$ divide $\gamma_0(p, q)$ on j curves $\gamma_0^{(i)}(p, q)$. Then there exist the similarity transformations $h_{0,i}$ with factors equal to powers of β such that $h_{0,i}[\gamma_0(p, q)] = \gamma_0^{(i)}(p, q)$.

Corollary. $\gamma_0(p, q)$ has partition into the sets similar to $\gamma_0(p, q)$:

$$\gamma_0(p, q) = \bigcup_{i=1}^{j} h_{0,i}(\gamma_0^{(i)}(p, q)). \tag{12}$$

So, in practice, we can find the decomposition (12). From this decomposition we can obtain vertices of polyline L, and, consequently, the map constr_L.

Table 5
Codes of oriented polylines L determining elementary boundaries $\gamma(p, q)$ for tilings from Q_3.

	p			
q	0	1	2	3
1	$3\bar{2}$	$3\bar{4}3$	–	–
2	$3\bar{2}\bar{2}$	$3\bar{2}3\bar{4}3$	$333\bar{4}\bar{1}$	–
3	$3\bar{2}\bar{2}\bar{2}$	$3\bar{2}3\bar{2}3\bar{4}3$	$3\bar{2}333\bar{4}\bar{1}$	$3333\bar{4}\bar{1}\bar{1}$
4	$3\bar{2}\bar{2}\bar{2}\bar{2}$	$3\bar{2}3\bar{2}3\bar{2}3\bar{4}3$	$3\bar{2}\bar{2}3333\bar{4}\bar{1}$	$\bar{2}3333\bar{4}3\bar{1}\bar{1}$
5	$3\bar{2}\bar{2}\bar{2}\bar{2}\bar{2}$	$3\bar{2}3\bar{2}3\bar{2}3\bar{2}3\bar{4}3$	$3\bar{2}\bar{2}\bar{2}33333\bar{4}\bar{1}$	$\bar{2}\bar{2}3333\bar{4}33\bar{1}\bar{1}$
	$3\bar{2}^q$	$3(\bar{2}3)^{q-1}\bar{4}3$	$3\bar{2}^{q-2}3^q\bar{4}\bar{1}$	$\bar{2}^{q-3}3^4\bar{4}3^{q-3}\bar{1}^2$

There exists an alternative approach for constructing the maps constr_L corresponding to fractal elementary boundaries $\gamma_0(p, q)$. This approach is based on the following conjecture.

Conjecture 5. Let l_i be a vector of some segment of the oriented polyline L corresponding to a map constr_L for the elementary boundary $\gamma_0(p, q)$. Then $l_i = \eta_i \beta^{-d_i} z$, where d_i is a natural number and $\eta_i = \pm 1$ and z is a complex number corresponding to the vector $\overrightarrow{Z_0 Z_1}$. Moreover, we have the equality

$$\sum \eta_i \beta^{-d_i} = 1, \qquad (13)$$

where the summation is taken over all segments of the oriented polyline L.

Note that equation (13) assumes an ordered sum; then the equation explicitly determines the polyline L. Moreover, the oriented polyline L can be specified as a sequence of pairs (η_i, d_i) in view of their order. For short we will write only values d_i for $\eta_i = 1$. We will also use the designation $\overline{d_i}$ in the case of $\eta_i = -1$. We will also use the designation d^k if the value d is repeated k times consecutively in the code of the polyline. Note that the expansion (13) can be obtained from the search of algebraic relations equivalent to an equation for β^{-1}. In Table 5 we represent codes of oriented polylines L for some tilings from Q_3. For $p = 0, 1, 2, 3$, we can write these codes for general q. For the set Q_4 and $a = 2$ the code of the oriented polyline is $4\overline{32}$. For the set Q_5 in the cases $a = 0$ and $a = 1$ the codes of oriented polylines are $5\bar{4}$ and $5\overline{441}$, respectively. For other values of a, codes of polylines for tilings from the sets Q_4 and Q_5 are not known. Note that by using Table 5 we can prove conjectures 3–5 for $p = 0, 1, 2, 3$.

Now we can describe parameters of boundaries of the tiling $\mathrm{Til}(p, q)$. Clearly, using an algorithm similar to algorithm 1, we can reduce the calculation of parameters of all boundaries to calculation of parameters of boundaries from the nuclear $\mathrm{Nucl}(p, q)$. Considering the similarity of all tiles it is sufficient to calculate the parameters $\Gamma(p, q)'$ of the boundary of one tile. In view of equations (9) and (10) it is sufficient to calculate parameters $\gamma_0(p, q)'$. However, the construction of all parameters from $\gamma_0(p, q)'$ is an extremely difficult problem. Nevertheless, it is possible to construct parameters of some point set, everywhere dense in $\gamma_0(p, q)$.

Let $\gamma_{0,n}^{(\mathrm{app})}$ be a set of vertices of oriented polyline $\mathrm{constr}_L^{(n)}(\overrightarrow{Z_0 Z_1})$ and

$$\gamma_0^{(\mathrm{app})} = \bigcup_{i=0}^{\infty} \gamma_{0,i}^{(\mathrm{app})}$$

[the superscript '(app)' stands for 'approximating point set']. It is obvious that $\gamma_0^{(\mathrm{app})}$ is everywhere dense in $\gamma_0(p, q)$. From the definition of the map constr_L we can obtain recurrent formulae expressing points from $\gamma_{0,n}^{(\mathrm{app})}$ through points from $\gamma_{0,i}^{(\mathrm{app})}$, $i < n$. So, all points from $\gamma_{0,n}^{(\mathrm{app})}$ rationally express through vertices Z_0 and Z_1. Using the conjugation map, we can transfer the recurrent formula to the set of parameters. So, we can call parameters for all points from $\gamma_0^{(\mathrm{app})}$. After that, we can describe parameters for all points from $\gamma_{\mathrm{all}}^{(\mathrm{app})}$. Here $\gamma_{\mathrm{all}}^{(\mathrm{app})}$ is the set of points corresponding to points from $\gamma_0^{(\mathrm{app})}$ on all boundaries of the tiles from $\mathrm{Til}(p, q)$.

Alternative parameterization of boundary $\gamma_0(p, q)$ is obtained by Messaoudi (2005). This approach does not use the conjugation map.

8. Similarity transformations of the tilings

Tilings $\mathrm{Til}(p, q)$ are not periodic. Therefore they cannot have translations in their symmetry groups. Instead of crystallographic symmetry transformations we can consider the similarity transformations of the tilings $\mathrm{Til}(p, q)$.

The transformation $h: z \rightarrow \alpha z + (1 - \alpha) c$ is called a similarity transformation of the tiling $\mathrm{Til}(p, q)$ if it maps the set of boundaries of the tiling $\mathrm{Til}(p, q)$ into itself. The complex number α is a similarity factor, c is a similarity center.

From the definition we immediately have the following properties of similarity transformations.

(i) Let h_1 and h_2 be two similarity transformations of the tiling $\mathrm{Til}(p, q)$. Then the composition $h_1 \circ h_2$ is well defined and this composition is a similarity transformation of the tiling $\mathrm{Til}(p, q)$.

(ii) Let $e: z \rightarrow z$ be the identical transformation. Then e is a similarity transformation of the tiling $\mathrm{Til}(p, q)$. For any similarity transformation h we have $h \circ e = e \circ h = h$.

(iii) Let $h \neq e$ be a similarity transformation of the tiling $\mathrm{Til}(p, q)$. Then transformation h^{-1} cannot be a similarity transformation of this tiling.

Let G be a set of all similarity transformations of the tiling $\mathrm{Til}(p, q)$. From property (iii) it immediately follows that the set G is not a group. Nevertheless, from properties (i) and (ii) and from associativity of composition it follows that the set G is semigroup with the composition operation.

Similarity of quasiperiodic tilings is a natural generalization of the similarity of model sets. Evaluation of the similarities of model sets can be found by Cotfas (1999).

Conjecture 6. Let h be a similarity transformation of tiling $\mathrm{Til}(p, q)$. Then its similarity factor is equal to $\pm \beta^s$ for some $s \in \mathbb{N}$.

Further, for simplicity, we consider only similarity transformations with the factor β^s. Denote by G^+ the set of such similarity transformations.

Similarity of the tilings $\mathrm{Til}(p, q)$ is studied by Zhuravlev & Maleev (2009*a*,*b*) in the special case of Rauzy tiling $\mathrm{Til}(1, 1)$. Thus the following facts were discovered.

Proposition 10. Let h be a similarity transformation with the factor β^s. Suppose that h maps the set of all vertices of the tiling Til(1, 1) to itself. Then h is a similarity transformation of the tiling Til(1, 1). The converse is also true.

Unfortunately, for arbitrary tilings Til(p, q), a similar proposition is incorrect. In the general case we have only the following result.

Proposition 11. Let h be a similarity transformation which maps the set $\gamma_{\rm all}^{\rm (app)}$ to itself. Then h is a similarity transformation of the tiling Til(p, q). The converse is also true.

Note that the conjugation map reduces the determination of the similarity transformations of the tiling Til(p, q) to determination of all transformations of the form $x \to \alpha' x + (1 - \alpha')c'$ which maps the set of parameters of boundaries to itself. Proposition 11 implies that it is sufficient to find the transformations which map the closure $\overline{\gamma_{\rm all}^{\rm (app)}(p, q)'}$ to itself. Recall that $\overline{\gamma_{\rm all}^{\rm (app)}(p, q)'}$ is a union of countable sets of segments. This approach can be used for a practical calculation of the similarity of the tiling Til(p, q).

The conditions of proposition 11 can probably be weaker. Let $\gamma_{{\rm all},n}^{\rm (app)}$ be a set of points from the boundaries of the tiling Til(p, q) corresponding to the points from $\gamma_{0,n}^{\rm (app)}$.

Conjecture 7. For any tiling Til(p, q) there exists a number n (depending on p, q) such that h is a similarity transformation of the tiling Til(p, q) if and only if h maps the set $\gamma_{{\rm all},n}^{\rm (app)}$ to itself.

Note that the map $z \to \beta^s z$ is a similarity transformation of the tiling Til(p, q).

Let G^0 be a set of generators of the semigroup G^+, *i.e.*

(i) Any similarity transformation from G^+ is a composition of the similarity transformations from G^0.

(ii) Any similarity transformation from G^0 cannot be represented as a composition of another similarity transformation from G^0.

Note that the similarity transformations with the factor β belong to G^0.

For the tiling Til(1, 1) in the paper by Zhuravlev & Maleev (2009*b*) it is shown that for every generator from G^0 the corresponding similarity center is a vertex of the tiling Til(1, 1) and its similarity factor is β. For arbitrary tilings Til(p, q) these results are false. Nevertheless we have the following conjecture.

Conjecture 8. Let $G_s^0(p, q)$ be a set of generators from G^0 with the factor β^s. Let $C(G^0)$ be a set of their similarity centers. Then the set $G_s^0(p, q)$ is empty for any $s \geq s_1(p, q)$. Here $s_1(p, q)$ is a constant which depends only on p and q.

9. Conclusions

We have considered a method of construction of quasiperiodic tilings Til(p, q) based on β-expansions corresponding to cubic irrationalities. We obtained three classes of tilings, Q_3, Q_4 and Q_5, which consisted of three, four and five pairwise similar prototiles, respectively. For all classes we obtained weak parameterization for the tilings. Thus we have a new convenient algorithm for construction of the tilings.

We also considered strong parameterization. This parameterization determines local tiles encirclements. Using strong parameterization we obtained polygonal layerwise growth of the tilings Til(p, q). In other words, there exists the polygon pol(p, q) such that

$$\lim_{n\to\infty} \frac{eq_n(P)}{n} = {\rm pol}(p, q),$$

where $eq_n(P)$ is the nth coordination encirclement defined above.

From the theorem of the complexity function we found that the n-crown of the nuclear Nucl(p, q) of the tiling determines the value of the complexity function $N_{{\rm Til}(p,q)}(n) = \#C_n[{\rm Nucl}(p, q)]$ and all possible variants of n-crown of tiles from Til(p, q).

It is established that the boundary of the tile $\Gamma(p, q)$ can be represented as a union of elementary fractal boundaries $\gamma_i(p, q)$. These elementary boundaries are pairwise similar. We also consider algorithms for calculating elementary fractal boundaries.

There are many similarity transformations that map boundaries of the tilings Til(p, q) to themselves. It is shown that the set of such transformations is a semigroup. It is possible to use parameters of vertices and boundaries of the tilings Til(p, q) to study this semigroup.

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